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Quantizing three-spin string solution in $AdS_5 \times S^5$

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Abstract: As was recently found in [3], there exists a simple classical solution describing a closed string rotating in S^5 and located at the center of AdS_5. It is parametrized by the angular momentum J of the center of mass and two equal SO(6) angular momenta J' in the two other orthogonal rotation planes. The corresponding dual $N = 4$ SYM operators should be scalar operators in SU(4) representations $[0, J - J', 2J']$ if $J \geq J'$, or $[J' - J, 0, J' + J]$ if $J' \geq J$. This solution is stable if $J' \leq \frac{3}{2}J$ and for large $J + 2J'$ its classical energy admits an expansion in positive powers of $\frac{\lambda}{(J+2J')^2}$ ($\sqrt{\lambda}$ is proportional to string tension): $E = J + 2J' + \frac{\lambda}{(J+2J')^2}J' + \cdots$. This suggests a possibility of a direct comparison with perturbative SYM results for the corresponding anomalous dimensions in the sector with $\frac{\lambda}{(J+2J')^2} \ll 1$, by analogy with the BMN case. We conjecture that all quantum sigma model string corrections are then subleading at large J', so that the classical formula for the energy is effectively exact to all orders in λ. It could then be interpolated to weak coupling, representing a prediction for the anomalous dimensions on the SYM side. We test this conjecture by computing the 1-loop superstring sigma model correction to the classical energy.

Keywords: Superstrings and Heterotic Strings, String Duality, AdS-CFT and dS-CFT Correspondence.

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1. Introduction

Motivated by attempts [1, 2] to extend AdS/CFT duality to non-BPS states we have recently proposed [3] to study the $AdS_5 \times S^5$ string — $N = 4$ SYM duality in a new sector parametrised by several components of S^5 spin or several "R-charges".

We have found a new classical solution describing a circular closed string located at the origin of AdS_5 space and rotating in S^5 with two equal angular momenta in the two orthogonal planes: the rotating string moves on S^3 within S^5 just as in the case of a two-spin flat-space solution where the string rotates in two orthogonal planes while always lying on a 3-sphere in R^4. In addition, the center of mass of the string may be rotating along another circle of S^5, leading to a particular string solution with all the three S^5 charges being non-zero ($J_1 = J,\ J_2 = J_3 = J'$). The point-like string case of [1] corresponds to the special case of $J' = 0,\ J \neq 0$ when the energy of unexcited string is $E = J$. In another special case of $J = 0,\ J' \neq 0$ when the string has maximal size (so that $2J' \geq \sqrt{\lambda}$) the energy turns out to depend on J' in a remarkably simple way: $E = \sqrt{(2J')^2 + \lambda}$. While this solution with $J = 0$ appears to be unstable, there is always a non-trivial region of stability when $J \neq 0$. As we shall show below, the solution is stable in the case which is the most interesting from the point of view of the AdS/CFT comparison — when both J and J' are large compared to $\sqrt{\lambda}$.

Let us start with a brief review of the basic features of this classical solution [3]. Written in terms of the AdS_5 time coordinate t and the angles of S^5 metric

$$(ds^2)_{S^5} = d\gamma^2 + \cos^2\gamma\, d\varphi_1^2 + \sin^2\gamma\,(d\psi^2 + \cos^2\psi\, d\varphi_2^2 + \sin^2\psi\, d\varphi_3^2)\,, \tag{1.1}$$

the solution is

$$t = \kappa\tau\,, \qquad \gamma = \gamma_0\,, \qquad \varphi_1 = \nu\tau\,, \qquad \varphi_2 = \varphi_3 = \mathrm{w}\tau\,, \qquad \psi = \mathrm{k}\sigma\,, \tag{1.2}$$

where $\kappa, \gamma_0, \nu,$ w are constants, k is an integer[1] and τ and $\sigma \in (0, 2\pi)$ are the world-volume (2-cylinder) coordinates. The equations of motion and the conformal gauge constraints imply

$$\mathrm{w}^2 = \nu^2 + \mathrm{k}^2\,, \qquad \kappa^2 = \nu^2 + 2\mathrm{k}^2 \sin^2\gamma_0\,, \tag{1.3}$$

so that there are only two continuous independent parameters — ν (or κ) and γ_0, and one discrete one — k.

The case of $\mathrm{k} = 0$ corresponds to the point-like solution considered in [1] and interpreted in the context of semiclassical approximation in [2, 4]. Then $\varphi_1 = \varphi_2 = \varphi_3 = \nu\tau$, and there is an $O(6)$ rotation that transforms this solution into a null geodesic along a canonical large circle of S^5.

In what follows we shall be mostly interested in the minimal-energy sector with $\mathrm{k} = 1$ but will keep the k dependence in some expressions for generality.

The non-zero SO(6) angular momentum components J_{MN} then are $J_1 = J_{12}$, $J_2 = J_{34} = J'$, $J_3 = J_{56} = J'$, where[2]

$$J = \sqrt{\lambda}\,\nu\,\cos^2\gamma_0\,, \qquad J' = \frac{1}{2}\sqrt{\lambda}\,\sqrt{\nu^2 + \mathrm{k}^2}\,\sin^2\gamma_0\,. \tag{1.4}$$

$\sqrt{\lambda} = R^2/\alpha'$ is the effective dimensionless string tension related to 't Hooft coupling. The classical energy $E = \sqrt{\lambda}\,\kappa = \sqrt{\lambda}\,\mathcal{E}_0(\nu, \gamma_0)$ is then $E = E(J, J', \lambda)$. It can be expressed in terms of the R-charge J' and the auxiliary "charge" $\mathcal{V} = \sqrt{\lambda}\,\nu = \frac{1}{\cos\gamma_0}\,J$:

$$E = \mathcal{V}\sqrt{1 + \frac{2\lambda\mathrm{k}^2}{\mathcal{V}^2}\left(1 - \frac{J}{\mathcal{V}}\right)}\,, \tag{1.5}$$

[1] The standard range of the angles $\psi, \varphi_2, \varphi_3$ to cover S^3 only once is $0 < \psi \leq \frac{\pi}{2}$, $0 < \varphi_2 \leq 2\pi$, $0 < \varphi_3 \leq 2\pi$: then the S^3 ($\gamma = \pi/2$) embedding coordinates $X = \cos\psi\, e^{i\varphi_2}$, $Y = \sin\psi\, e^{i\varphi_3}$ have positive "radial" factors (their sign change can be compensated by $\varphi_{2,3} \to \varphi_{2,3} + \pi$). However, in the present case it is useful to choose a different range: $0 < \psi \leq 2\pi$, $0 < \varphi_2 \leq \pi$, $0 < \varphi_3 \leq \pi$. Then the constant φ_2, φ_3 section of S^3 will be the full circle (instead of its $(0, \frac{\pi}{2})$ quarter), and thus we will have a consistent map of the closed string into S^3 for *each* given moment of time τ, as required by the closed string interpretation.

[2] Here it is assumed that $\mathrm{k} \neq 0$; otherwise J' is to be multiplied by 2.

where $\mathcal{V} = \mathcal{V}(J, J', \lambda)$ is a solution of the (quartic) equation

$$\mathcal{V} = J + \frac{2J'}{\sqrt{1 + \lambda \mathrm{k}^2/\mathcal{V}^2}} . \qquad (1.6)$$

Note that at the classical level the dependence on k can be absorbed into the string tension $\sqrt{\lambda}$.

As we shall see below, at large ν, this solution is stable under small perturbations provided

$$J' \leq \frac{4\mathrm{k} - 1}{2(2\mathrm{k} - 1)^2} J , \quad \text{i.e.} \quad \frac{J'}{J + 2J'} \leq \frac{4\mathrm{k} - 1}{8\mathrm{k}^2} , \qquad (1.7)$$

or, for $\mathrm{k} = 1$,

$$J' \leq \frac{3}{2} J , \quad \text{i.e.} \quad \frac{J'}{J + 2J'} \leq \frac{3}{8} . \qquad (1.8)$$

In the limit we will be interested in here when $\nu \gg \mathrm{k}$ (similar to the limit considered in [1]), i.e. when $\frac{\sqrt{\lambda}\,\mathrm{k}}{\mathcal{V}} \ll 1$, the expression for the energy may be formally written as an expansion in positive powers of λ. More precisely, in the semiclassical approximation one has, of course, $\lambda \gg 1$, but the expansion is in $\frac{\lambda \mathrm{k}^2}{\mathcal{V}^2} \ll 1$.

Indeed, for large $J + 2J'$ we can find $\mathcal{V}$ from (1.6) as a series in $\frac{\lambda \mathrm{k}^2}{(J+2J')^2}$, i.e.

$$\mathcal{V} = J + 2J' - \frac{\lambda \mathrm{k}^2 J'}{(J + 2J')^2} + \cdots , \qquad (1.9)$$

so that

$$E = J + 2J' + \frac{\lambda\, \mathrm{k}^2 J'}{(J + 2J')^2} + \cdots , \qquad (1.10)$$

where the only requirement on J and J' is that $J + 2J' \gg \sqrt{\lambda}\ \mathrm{k}$. In addition to ν or $J + 2J'$, the classical energy and quantum corrections to it depend also on another parameter — γ_0, which in the $\nu \gg \mathrm{k}$ limit is given by (cf. (1.4), (1.9))

$$\sin^2 \gamma_0 = \frac{2J'}{J + 2J'} + \cdots . \qquad (1.11)$$

Note also that for $J \gg J'$ the energy (1.10) becomes

$$E = J + 2J' \left(1 + \frac{\lambda\, \mathrm{k}^2}{2J^2} + \cdots \right) , \qquad (1.12)$$

which is consistent with the string oscillation spectrum in the sector with large $J \gg \sqrt{\lambda}\,\mathrm{k}$ [1, 5], i.e. with the "plane-wave" [6] spectrum, where J' represents the angular momentum carried by string oscillations (similar expression is found if J' is replaced by spin in AdS_5 directions).

In [3] it was suggested that the corresponding dual $\mathcal{N} = 4$ SYM operators should be of the form $\mathrm{tr}[(\Phi_1 + i\Phi_2)^J (\Phi_3 + i\Phi_4)^{J'} (\Phi_5 + i\Phi_6)^{J'}] + \cdots$, where dots stand for appropriate permutations of all $J = 2J'$ scalar factors. These operators belong to the irreducible representation of SU(4) with Young tableau labels (J, J', J') or with Dynkin labels $[0, J - J', 2J']$ if $J \geq J'$, and to the representation (J', J', J) or $[J' - J, 0, J' + J]$ if $J' \geq J$. It is worth noting that due to the stability requirement (1.7), the operators dual to solutions with $\mathrm{k} \geq 2$ always have $J > J'$, and, therefore, belong to the representation $[0, J - J', 2J']$. We do not know if the solution with $\mathrm{k} = 1$ should be dual to an operator having minimal canonical dimension for given values of R-charges J and J'.[3] There might exist a more complicated solution with less energy describing, for example, a rotating folded string lying entirely in S^5.

In the large-N SYM perturbation theory ($\lambda \ll 1$) one expects to find corrections to the canonical dimension of these operators behaving as

$$\Delta(J, J', \lambda)_{\lambda \ll 1} = J + 2J' + \lambda F_1(J, J') + O(\lambda^2) \, . \tag{1.13}$$

The semiclassical result (1.10) is a prediction for the anomalous dimensions in the opposite $\lambda \gg 1$ limit when $J + 2J' \gg 1$. The dependence of the energy (1.10) on k may be reflecting a band structure of anomalous dimensions of the SYM side.[4]

Given a simple dependence of the energy E in (1.10) on λ in the limit $J + 2J' \gg \sqrt{\lambda}\, \mathrm{k}$, it was conjectured in [3] that the expression (1.10) may be valid also at *small* values of λ if $J + 2J'$ is very large. More explicitly, one may expect that the general expression for the anomalous dimension valid for any λ but with $J + 2J' \gg \sqrt{\lambda}\, \mathrm{k}$ (i.e. $\frac{\lambda \mathrm{k}^2}{(J+2J')^2} \ll 1$) is

$$\Delta(J, J', \lambda)_{J+2J' \gg \sqrt{\lambda}\, \mathrm{k}} = J + 2J' + \; f_1(J') \frac{\lambda \mathrm{k}^2}{(J + 2J')^2} + O\left(\frac{\lambda^2 \mathrm{k}^4}{(J + 2J')^4} \right), \tag{1.14}$$

where in the string perturbation theory limit ($J' \sim \sqrt{\lambda} \gg 1$)

$$f_1(J') = J' + b_1 + O\left(\frac{1}{J'} \right). \tag{1.15}$$

Similar expression is then expected for $\lambda \ll 1$, i.e. in the SYM perturbation theory.

In addition to the limit $J + 2J' \gg \sqrt{\lambda} \gg 1$, another special case is $J \gg J'$, where one may relate the resulting expression for the energy to the (non-perturbative) corrections to the dimensions of particular operators in the sector studied in [1].

On general grounds, the string sigma model corrections to the classical energy will have the following structure

$$E = \sum_{l=0}^{\infty} E_l = \sqrt{\lambda}\; \mathcal{E}_0(\nu, \gamma_0) + \mathcal{E}_1(\nu, \gamma_0) + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2(\nu, \gamma_0) + \cdots , \tag{1.16}$$

where $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots$ depend only on the parameters ν, γ_0 (and k) of the classical solution and should have, for $\nu \gg \mathrm{k}$, an expansion in inverse powers of ν, i.e. in powers of $\frac{\sqrt{\lambda}\, \mathrm{k}}{J+2J'}$.

[3] The same formula (1.10) should be giving the conformal dimensions of the operators from the two different ($[0, J - J', 2J']$ or $[J' - J, 0, J' + J]$) representations. This should be true not only in the large λ limit but also in the weak-coupling perturbation theory.

[4] These string solutions suggest that operators in these representations may be divided into different sectors parametrized by the integer k. In each sector there is an operator with the lowest conformal dimension that should be dual to the string solution, with other operators in the sector dual to excitations near that classical string solution.

For example,

$$\mathcal{E}_0(\nu,\gamma_0) = \kappa = \nu + \frac{1}{\nu}\sin^2\gamma_0 + O\left(\frac{1}{\nu^2}\right), \tag{1.17}$$

$$\mathcal{E}_1 = \sum_{m=-1}^{\infty} \frac{1}{\nu^m} e_m(\gamma_0)\,, \qquad e_m = c_m + d_m \sin^2\gamma_0 + \cdots . \tag{1.18}$$

Expressing this in terms of $J+2J' \gg 1$ and $\frac{J'}{J+2J'}$ using (1.4), (1.11) we find that the correction would be of the form (1.14), (1.15) if: (i) the functions $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \ldots$ vanish in the limit $\nu \to \infty$; (ii) the expansion of $\mathcal{E}_1$ goes over only even powers of $1/\nu$ and starts with $1/\nu^2$; the expansion of $\mathcal{E}_2$ goes over only odd powers of $1/\nu$ and starts with $1/\nu^3$, etc. In particular, the nonrenormalization of the leading $J+2J'$ term in E requires the vanishing the first three terms ($m=-1,0,1$) in (1.18).[5] In that case all string corrections can be written as functions of J, J' and λ with λ entering only in *positive* powers. This is very similar to what was found in the case of $\nu \neq 0$, $\mathrm{k}=0$ [3], i.e. in the BMN case. Assuming that the expansion of $\mathcal{E}_1$ starts with the $1/\nu^2$ term (as we shall indeed confirm below), one concludes that the coefficient b_1 in (1.15) is subleading at large J', and is equal to $e_2(\gamma_0)$ in (1.18). Indeed, we then have

$$\mathcal{E}_1 = \frac{1}{\nu^2}e_2(\gamma_0) + \frac{1}{\nu^4}e_4(\gamma_0) + \cdots = \bar{e}_2\left(\frac{J'}{J}\right)\frac{\lambda \mathrm{k}^2}{(J+2J')^2} + \bar{e}_4\left(\frac{J'}{J}\right)\frac{\lambda^2\mathrm{k}^4}{(J+2J')^4} + \cdots . \tag{1.19}$$

Similarly, we expect that higher-order terms in (1.16) will have the structure

$$\begin{aligned} E_l &= \frac{1}{(\sqrt{\lambda}\,)^{l-1}}\mathcal{E}_l(\nu,\gamma_0) = \frac{1}{(\sqrt{\lambda}\,\nu)^{l-1}}\left[\frac{1}{\nu^2}e_{2l}(\gamma_0) + \frac{1}{\nu^4}e_{4l}(\gamma_0) + \cdots\right] \\ &= \frac{1}{(J+2J')^{l-1}}\left[\bar{e}_{2l}\left(\frac{J'}{J}\right)\frac{\lambda\mathrm{k}^2}{(J+2J')^2} + \bar{e}_{4l}\left(\frac{J'}{J}\right)\frac{\lambda^2\mathrm{k}^4}{(J+2J')^4} + \cdots\right]. \end{aligned} \tag{1.20}$$

Thus, we expect that for large J and J' the classical expression for the energy (1.5) will be giving the leading contribution at any order in λ, and thus should be representing also the expression of the conformal dimension of the dual SYM operator computed in a weak-coupling expansion.

That would mean, in particular, that the leading one-loop perturbative correction to the dimension of the corresponding CFT operator dual to the string solution should indeed be of order $\frac{J'}{(J+2J')^2}$ and not of order $J+2J'$. The non-renormalization of the leading J-term in $E=\Delta$ does take place for the ground state in the BMN ($J'=0$) case, where one expands near a point-like BPS state. In the present case of extended rotating string solution the space-time supersymmetry is broken, but one may expect that it is in some sense "restored" in the limit $J+2J' \gg \sqrt{\lambda}\,\mathrm{k}$ (when a closed rotating string is moving fast along a circle in S^5); that should then be an explanation for the non-renormalization of the leading $J+2J'$ term in the energy.

[5]Given that for $\nu \gg \mathrm{k}$ one has $\frac{\sqrt{\lambda}\,\mathrm{k}}{J+2J'} \sim \frac{\mathrm{k}}{\nu} \ll 1$ as another small parameter of semiclassical expansion (in addition to $\frac{1}{\sqrt{\lambda}}$), one may expect, by analogy with the reasoning in [3, 7], that the leading $J+2J'$ term in E will not be renormalized to all orders in $\frac{1}{\sqrt{\lambda}}$. Corrections may be suppressed in the large 2-d mass (large ν) limit on a 2-d cylinder. Note that all parameters in (1.2), (1.3) scale as ν at large ν and there should be no parameter-independent constant terms in $\mathcal{E}_i$ because of supersymmetry.

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After a review of classical solution in terms of the embedding coordinates in section 2, we shall derive the general expression for the 1-loop corrections to the energy (1.10) coming both from the bosons and the Green-Schwarz fermions (section 3). We shall find the quartic equations for the (squares of) characteristic bosonic and fermionic frequencies. In section 4 we shall study the large ν (or, equivalently, large κ) limit of the 1-loop correction and confirm the structure of the 1-loop correction (1.19), computing the value of $e_2(\gamma_0)$. Some technical details will be given in appendices A,B and C.

The relevant bosonic and fermionic quadratic fluctuation parts of the Green-Schwarz $AdS_5 \times S^5$ action were already presented in [3] and will be reviewed and further simplified below. In spite of the classical solution being dependent on τ and σ, the quadratic fluctuation action can be put (after natural local "rotations" of fluctuation fields) in the form where it describes a collection of bosons and fermions in flat 2 dimensional space all having constant masses and coupled to constant (non)abelian 2-d gauge terms.[6] Remarkably, the form of this action is essentially the same as of the light-cone gauge superstring action in a particular plane-wave background with an antisymmetric 2-form field. The problem of finding the leading correction to the ground state energy and also of the spectrum of string excitations near the three-spin solution is thus closely related to the corresponding problem in the case of the "homogeneous" plane-wave backgrounds (cf. [8, 9]).

2. Classical solution and bosonic part of quadratic fluctuation action

Written in terms of the 6 embedding coordinates of S^5 into R^6 (here we rename the coordinates $5,6 \to 1,2$) $(X_1^2 + \cdots + X_6^2 = 1)$ [3]

$$X = X_1 + iX_2 = \sin\gamma \ \cos\psi \, e^{i\varphi_2}\,, \qquad Y = X_3 + iX_4 = \sin\gamma \ \sin\psi \, e^{i\varphi_3}\,, \tag{2.1}$$

$$Z = X_5 + iX_6 = \cos\gamma \ e^{i\varphi_1}\,, \tag{2.2}$$

the solution (1.2) is

$$X = \sin\gamma_0 \ \cos \mathrm{k}\sigma \ e^{i\mathrm{w}\tau}\,, \qquad Y = \sin\gamma_0 \ \sin \mathrm{k}\sigma \ e^{i\mathrm{w}\tau}\,, \qquad Z = \cos\gamma_0 \ e^{i\nu\tau}\,. \tag{2.3}$$

The conformal gauge constraints are then satisfied provided the coefficient κ of the AdS_5 coordinate t is related to the parameters as in (1.3).

This solution can be found directly by starting with the S^5 (or $O(6)$) sigma model action in conformal gauge (Λ is a Lagrange multiplier field, and $\eta_{ab} = (-,+)$)

$$I_{S^5} = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \ L\,, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}\,,$$
$$L = -\partial_a X_m \partial^a X_m - \Lambda (X_m X_m - 1)\,, \qquad m = 1, \ldots, 6\,. \tag{2.4}$$

The classical equations of motion then are

$$(-\partial^2 + \Lambda) X_m = 0\,, \qquad X_m X_m = 1\,, \qquad \Lambda = -\partial_a X_m \partial^a X_m\,. \tag{2.5}$$

[6] The constant connection terms can be eliminated at the expense of making the mass terms non-diagonal and τ and σ-dependent.

They are satisfied by (2.3) with

$$\mathrm{w}^2 = \nu^2 + \mathrm{k}^2 \,, \qquad \Lambda = \nu^2 \,. \tag{2.6}$$

This is an example of a special class of simple solutions of the non-linear equations (2.5) for which Λ = const and $X_m(\tau,\sigma)$ can be represented as a product of commuting $O(6)$ transformations depending on τ or σ and applied to a constant unit 6-vector. Indeed, we can write (2.3) as ($\mathrm{X} \equiv (X_m)$)

$$\mathrm{X}(\tau,\sigma) = O_{12+34}(\mathrm{w}\tau)\ O_{13+24}(\mathrm{k}\sigma)\ O_{56}(\nu\tau)\ O_{15}(\gamma_0)\ \mathrm{X}_0 \,, \qquad \mathrm{X}_0 = (1,0,0,0,0,0) \,, \tag{2.7}$$

where

$$O_{pq}(\alpha) \;=\; e^{-\alpha \mathcal{I}_{pq}} = I + \mathcal{I}^2_{pq}(1-\cos\alpha) - \mathcal{I}_{pq}\sin\alpha \tag{2.8}$$

$$O_{pq+kl}(\alpha) \;=\; e^{-\alpha(\mathcal{I}_{pq}+\mathcal{I}_{kl})} \,, \qquad (\mathcal{I}_{pq})_{mn} = \delta_{pm}\delta_{qn} - \delta_{pn}\delta_{qm} \,, \tag{2.9}$$

and $\mathcal{I}_{pq}$ is a generator of rotation in the (pq) plane in the fundamental representation of $O(6)$. Note that $[\mathcal{I}_{12}+\mathcal{I}_{34}, \mathcal{I}_{13}+\mathcal{I}_{24}] = 0$, $[\mathcal{I}_{12}+\mathcal{I}_{34}, \mathcal{I}_{56}] = 0$, $[\mathcal{I}_{13}+\mathcal{I}_{24}, \mathcal{I}_{56}] = 0$, i.e. the non-constant rotations commute. This simplifies dramatically the form of the small-fluctuation action.

In general, the quadratic fluctuations near a solution of (2.5) are described by ($X_m \to X_m + \tilde{X}_m$, $\Lambda \to \Lambda + \tilde{\Lambda}$)

$$L_2 = -\partial_a \tilde{X}_m \partial^a \tilde{X}_m - \Lambda \tilde{X}_m \tilde{X}_m - 2\tilde{\Lambda} X_m \tilde{X}_m \,, \tag{2.10}$$

i.e. they satisfy

$$[\delta_{mn} - X_m X_n](-\partial^2 \tilde{X}_n + \Lambda \tilde{X}_n) \;=\; 0 \,, \tag{2.11}$$

$$X_m \tilde{X}_m \;=\; 0 \,. \tag{2.12}$$

To find the action for the independent 5 fluctuations we are thus to solve the constraint (2.12) and substitute the result into the "unconstrained" action

$$\tilde{L}_2 = -\partial_a \tilde{X}_m \partial^a \tilde{X}_m - \Lambda \tilde{X}_m \tilde{X}_m \,. \tag{2.13}$$

Finally, one may solve the (relevant linear part of) the conformal gauge constraints,

$$-\kappa\partial_\tau \tilde{t} + \partial_\tau X_m \partial_\tau \tilde{X}_m + \partial_s X_m \partial_s \tilde{X}_m = 0 \,, \qquad -\kappa\partial_\sigma \tilde{t} + \partial_\tau X_m \partial_\sigma \tilde{X}_m = 0 \,, \tag{2.14}$$

but this is not necessary in order to determine the non-trivial part of the spectrum. In addition, one needs to include the contribution of 4 massive bosonic fluctuations in the AdS_5 directions [3]

$$L_{AdS_5} = -\partial_a y_l \partial^a y_l - \kappa^2 y_l y_l \,, \qquad l = 1,2,3,4 \,. \tag{2.15}$$

In the present case of the solution (2.7), (2.6) it is easy to solve (2.12) (i.e. $\mathrm{X}^T\tilde{\mathrm{X}} = 0$) by a field redefinition on $\tilde{X}_m$ that "undoes" the rotation in (2.7), i.e.

$$\tilde{\mathrm{X}}(\tau,\sigma) = O_{12+34}(\mathrm{w}\tau)\ O_{13+24}(\mathrm{k}\sigma)\ O_{56}(\nu\tau)\ O_{15}(\gamma_0)\ \bar{\mathrm{X}}(\tau,\sigma) \,, \tag{2.16}$$

where $\bar{X}_m$ are the new ("tangent-space") fluctuations now subject to the simple (τ,σ)-independent condition $\bar{X}_m(X_m)_0 = 0$, which is solved by setting

$$\bar{X}_1 = 0 \,. \tag{2.17}$$

2.1 Fluctuation lagrangian

Equivalently, that means, combining (2.7), (2.16), (2.17), that we parametrize the classical+fluctuation field as

$$\mathrm{X}(\tau,\sigma) = O_{12+34}(\mathrm{w}\tau)\ O_{13+24}(k\sigma)\ O_{56}(\nu\tau)\ O_{15}(\gamma_0)\ \hat{\mathrm{X}}(\tau,\sigma)\,,$$
$$\hat{\mathrm{X}} = (1,\bar{X}_2,\bar{X}_3,\bar{X}_4,\bar{X}_5,\bar{X}_6)\,. \tag{2.18}$$

Doing the transformation (2.16) in (2.13) one ends up with the following simple fluctuation lagrangian with *constant* coefficients (determined essentially by the generators of $O(6)$ rotations in the classical solution):

$$\begin{aligned}\tilde{L}_2 =\ & (\partial_\tau \bar{X}_m)^2 - (\partial_\sigma \bar{X}_m)^2 + \ 4\nu(\cos\gamma_0 \bar{X}_1 + \sin\gamma_0\ \bar{X}_5)\partial_\tau \bar{X}_6 - \\ & -4\mathrm{w}[(\cos\gamma_0\ \bar{X}_5 - \sin\gamma_0 \bar{X}_1)\partial_\tau \bar{X}_2 + \bar{X}_3\partial_\tau \bar{X}_4] + \\ & + 4\mathrm{k}[(\cos\gamma_0\ \bar{X}_5 - \sin\gamma_0\ \bar{X}_1)\partial_\sigma \bar{X}_3 - \bar{X}_2\partial_\sigma \bar{X}_4]\,,\end{aligned} \tag{2.19}$$

where we have used integration by parts. Here the dependence on γ_0 could be rotated away if not for the constraint (2.17) we still need to impose. As a result, the fluctuation lagrangian for the 5 independent fluctuation fields becomes ($s = 2,3,4,5,6$)

$$\begin{aligned}L_2 =\ & (\partial_\tau \bar{X}_s)^2 - (\partial_\sigma \bar{X}_s)^2 + 4\nu\sin\gamma_0\ \bar{X}_5\partial_\tau \bar{X}_6 - 4\mathrm{w}(\cos\gamma_0\ \bar{X}_5\partial_\tau \bar{X}_2 + \bar{X}_3\partial_\tau \bar{X}_4) + \\ & +4\mathrm{k}(\cos\gamma_0\ \bar{X}_5\partial_\sigma \bar{X}_3 - \bar{X}_2\partial_\sigma \bar{X}_4)\,.\end{aligned} \tag{2.20}$$

The point-like (BMN) limit corresponds to the case of $\mathrm{k} = 0$ (then $\mathrm{w} = \nu$). The resulting lagrangian can be shown to be equivalent to the one found by expanding near a "canonical" BMN solution $L_2 = -(\partial_a \bar{X}_5)^2 - (\partial_a \tilde{X}_i)^2 - \nu^2 \tilde{X}_i^2$, where $i = 1,2,3,4$ (with the constraint (2.17) now being $\bar{X}_6 = 0$).

Eq. (2.20) is a special case of the following 2-d lagrangian

$$L = (\partial_\tau x_p)^2 - (\partial_\sigma x_p)^2 + 2f_{pq}x_p\partial_\tau x_q - 2h_{pq}x_p\partial_\sigma x_q\,, \tag{2.21}$$

where f_{pq} and h_{pq} are constant antisymmetric coefficient matrices. The latter can be written also as (ignoring total derivative)

$$L = (\partial_\tau x_p + f_{pq}x_q)^2 - (\partial_\sigma x_p + h_{pq}x_q)^2 - (f_{pq}f_{kq} - h_{pq}h_{kq})x_p x_k\,, \tag{2.22}$$

i.e. it represents a massive scalar 2-d theory coupled to a constant 2-d gauge field (which can be "rotated away" at the expense of making the mass term τ and σ dependent). The corresponding hamiltonian is

$$H = (\partial_\tau x_p)^2 + (\partial_\sigma x_p + h_{pq}x_q)^2 + (f_{pq}f_{kq} - h_{pq}h_{kq})x_p x_k\,. \tag{2.23}$$

In the case of (2.20) we find

$$\begin{aligned}H_2 =\ & (\partial_\tau \bar{X}_2)^2 + (\partial_\tau \bar{X}_3)^2 + (\partial_\tau \bar{X}_4)^2 + (\partial_\tau \bar{X}_5)^2 + (\partial_\tau \bar{X}_6)^2 + (\partial_\sigma \bar{X}_2 - \mathrm{k}\bar{X}_4)^2 + \\ & +(\partial_\sigma \bar{X}_4 + \mathrm{k}\bar{X}_2)^2 + (\partial_\sigma \bar{X}_3 - \mathrm{k}\cos\gamma_0 \bar{X}_5)^2 + (\partial_\sigma \bar{X}_5 + \mathrm{k}\cos\gamma_0\ \bar{X}_3)^2 + \nu^2(\bar{X}_5^2 + \bar{X}_4^2) + \\ & +(\nu^2 + \mathrm{k}^2\sin^2\gamma_0)\bar{X}_3^2 + (\nu\sin\gamma_0 \bar{X}_6 - \mathrm{w}\cos\gamma_0 \bar{X}_2)^2 - \mathrm{k}^2\bar{X}_2^2\,.\end{aligned} \tag{2.24}$$

While the hamiltonian is always positive in the point-like case $\mathrm{k} = 0$, it is not manifestly so for $\mathrm{k} > 0$, i.e. there is a potential for an instability. In general, the conclusion about non-positivity and instability is not directly obvious on a cylinder; for example, we cannot set $\partial_\sigma \bar{X}_4 + \mathrm{k}\bar{X}_2 = 0$ for constant $\bar{X}_2$ since then $\bar{X}_4$ will not be periodic in σ. The instability is always present when $\gamma_0 = \pi/2$, i.e. when $\bar{X}_6$ and $\bar{X}_2$ are decoupled. As we shall argue below, there is a range of parameters (large ν and sufficiently small γ_0) for which the solution is stable under small perturbations.

Let us note that the lagrangian (2.23) can be also interpreted as a light-cone gauge ($u = \tau$) lagrangian for the bosonic string sigma model $L = -(\eta^{ab} g_{mn} + \epsilon^{ab} B_{mn})\partial_a x^m \partial_b x^n$ in (in general, non-conformal) plane-wave background with the following metric and the antisymmetric 2-form field (cf. [9])

$$ds^2 = 2dudv + 2f_{pq}x_p dx_q du + dx_p dx_p\,, \qquad B_2 = 2h_{pq}x_p dx_q \wedge du\,. \tag{2.25}$$

The general form of (the linear part of) the conformal gauge constraints is

$$\begin{aligned}
\kappa\partial_\tau \tilde{t} &= \kappa^2 \bar{X}_1 - \mathrm{k}^2 \sin 2\gamma_0\ \bar{X}_5 + \mathrm{k}\sin\gamma_0\ \partial_\sigma \bar{X}_3 + \mathrm{w}\sin\gamma_0\ \partial_\tau y_1 + \nu\cos\gamma_0\ \partial_\tau \bar{X}_6 \\
\kappa\partial_s \tilde{t} &= 2\mathrm{kw}\sin\gamma_0\ \bar{X}_4 + \mathrm{k}\sin\gamma_0\ \partial_\tau \bar{X}_3 + \mathrm{w}\sin\gamma_0\ \partial_\sigma y_2 + \nu\cos\gamma_0\ \partial_\sigma \bar{X}_6\,.
\end{aligned} \tag{2.26}$$

Adding $-\partial_a \tilde{t}\partial^a \tilde{t}$ term and eliminating t from the resulting action implies cancellation also of one ("massless") combination of $\bar{X}_s$ coordinates; after a field redefinition one ends up with the following "reduced" lagrangian for the remaining 4 non-trivial fluctuations:

$$\begin{aligned}
L_2' &= (\partial_\tau \bar{X}_s)^2 - (\partial_\sigma \bar{X}_s)^2 - 4\nu C_1 \bar{X}_5 \partial_\tau \bar{X}_6 - 4\kappa C_2 \bar{X}_5 \partial_\tau \bar{X}_2 + 4\kappa C_1 \bar{X}_4 \partial_\sigma \bar{X}_2 + 4\nu C_2 \bar{X}_4 \partial_\sigma \bar{X}_6\,, \\
C_1 &= \left[\frac{\mathrm{k}^2(1+\sin^2\gamma_0)}{\nu^2 + \mathrm{k}^2\sin^2\gamma_0}\right]^{1/2}, \qquad C_2 = \left[\frac{\nu^2 + \mathrm{k}^2}{\nu^2 + \mathrm{k}^2\sin^2\gamma_0}\right]^{1/2}.
\end{aligned} \tag{2.27}$$

In the special case of $\nu = 0$ this is equivalent to the fluctuation lagrangian obtained in [3] in the static gauge.

2.2 Characteristic frequences

To find the spectrum of characteristic frequencies corresponding to the action (2.10) (for a general analysis of the theory (2.23) see [9]) we note that $\bar{X}_s$ fields must be periodic in σ (the rotations (2.16) we made preserve the periodicity) so that one can expand the solution of the quadratic fluctuation equations in modes

$$\bar{X}_s = \sum_{n=-\infty}^{\infty} \sum_{i=1}^{8} A_{sn}^{(i)} e^{i(\omega_{n,i}\tau + n\sigma)}\,, \tag{2.28}$$

where i labels different frequencies for a given value of n (we shall often suppress the index i below). Plugging this into the classical equations that follow from (2.10) one finds the following result for the determinant of the characteristic matrix

$$\begin{aligned}
\det M &= -(n^2 - \Omega) B_8(\Omega)\,, \qquad \Omega \equiv \omega_n^2\,, \\
B_8(\Omega) &= \Omega^4 + (6\mathrm{k}^2 - 8\nu^2 - 4n^2 - 2\mathrm{k}^2 \cos 2\gamma_0)\Omega^3 +
\end{aligned}$$

$$
\begin{aligned}
&+\Big[8\mathrm{k}^4+6\mathrm{k}^2n^2+6n^4+24\mathrm{k}^2\nu^2+16n^2\nu^2+16\nu^4+\\
&\quad+(8\mathrm{k}^4+2\mathrm{k}^2n^2+8\mathrm{k}^2\nu^2)\cos 2\gamma_0\Big]\Omega^2+\\
&+\Big[-16\mathrm{k}^4n^2+6\mathrm{k}^2n^4-4n^6-8\mathrm{k}^2n^2\nu^2-8n^4\nu^2+\\
&\quad+(-16\mathrm{k}^4n^2+2\mathrm{k}^2n^4-24\mathrm{k}^2n^2\nu^2)\cos 2\gamma_0\Big]\Omega+\\
&+n^4(n^2-4\mathrm{k}^2)(n^2-2\mathrm{k}^2-2\mathrm{k}^2\cos 2\gamma_0)\,. \qquad (2.29)
\end{aligned}
$$

Setting $\det M=0$, we observe the existence of one decoupled massless scalar field corresponding to the solution $\Omega=n^2$ of (2.29). The decoupled massless scalar is a reflection of the conformal gauge choice we made. We also find a nontrivial quartic equation for the remaining modes, giving 4 (in general, different) values for $|\omega_n|$, i.e. 8 characteristic frequencies $\omega_{n,i}$. Here k can be set to 1 (it can be restored by $n\to n/\mathrm{k},\ \nu\to\nu/\mathrm{k},\ \omega_n\to\omega_n/\mathrm{k}$).

The same result for the characteristic determinant (but without the zero-mode factor) is found by starting from the "reduced" action (2.27).

The BMN limit corresponds to setting $\mathrm{k}=0$ in the above formulae; while the fluctuation lagrangian (2.20) seems to depend on γ_0, the spectrum, as one might expect, does not: for $\mathrm{k}=0$ the determinant (2.29) becomes

$$
\mathrm{k}=0:\qquad B_8=(n^2-2\nu\omega_n-\omega_n^2)^2(n^2+2\nu\omega_n-\omega_n^2)^2\,, \qquad (2.30)
$$

and thus the roots are

$$
\omega_n=\pm\nu\pm\sqrt{n^2+\nu^2}\,. \qquad (2.31)
$$

Here the linear ν terms reflect the rotation of the fluctuations made in (2.16) while $\pm\sqrt{n^2+\nu^2}$ are the standard "plane-wave" frequencies. Similar result is found also in the fermionic sector discussed in the next section.

Another special case is when γ_0 is approaching $\frac{\pi}{2}$

$$
\gamma_0=\frac{\pi}{2}:\qquad J=0\,,\qquad J'\neq 0\,,\qquad E=2J'\sqrt{1+\frac{\lambda\mathrm{k}^2}{(2J')^2}}\,. \qquad (2.32)
$$

Setting $\gamma_0=\pi/2$ in (2.29) and solving $B_8=0$ we find the following 4+4 frequencies

$$
\omega_n^2=n^2+2(\nu^2+\mathrm{k}^2)\pm 2\sqrt{(\nu^2+\mathrm{k}^2)^2+n^2(\nu^2+2\mathrm{k}^2)}\,, \qquad (2.33)
$$

$$
\omega_n=\pm\nu\pm\sqrt{n^2+\nu^2}\,. \qquad (2.34)
$$

The condition of reality of ω_n in (2.33) is $n^2(n^2-4\mathrm{k}^2)\geq 0$, i.e. this solution has unstable modes with $n=\pm1,\ldots,\pm(2\mathrm{k}-1)$, as was already found in [3] for $\mathrm{k}=1$.

$J=0$ is found also when $\nu=0$ (see (1.4)); in this case

$$
\nu=0:\qquad \mathcal{J}'=\frac{1}{2}\sqrt{\lambda}\ \mathrm{k}\ \sin^2\gamma_0\,,\qquad J=0\,,\qquad E=\sqrt{4\sqrt{\lambda}\ \ \mathrm{k}\ J'}\,, \qquad (2.35)
$$

i.e. here J' bounded from above for fixed integer k ($\mathrm{k}=1$ case was discussed in [3]). Here the non-trivial characteristic frequences are

$$
\omega_n^2=n^2+2\mathrm{k}^2(2-\sin^2\gamma_0)\pm 2\mathrm{k}\sqrt{2n^2(2-\sin^2\gamma_0)+\mathrm{k}^2\sin^2\gamma_0}\,. \qquad (2.36)
$$

The condition of reality of ω_n here is $(n^2-4\mathrm{k}^2)(n^2-4\mathrm{k}^2+4\mathrm{k}^2\sin^2\gamma_0)\geq 0$, which is satisfied for $\sin^2\gamma_0 \leq \frac{4\mathrm{k}-1}{4\mathrm{k}^2}$, i.e. $J' \leq \sqrt{\lambda}\,\frac{4\mathrm{k}-1}{8\mathrm{k}}$. The same stability condition was found in [3] in the special case of $\mathrm{k}=1$. Thus for $\nu=0$ the combination $\frac{\lambda}{J'^2}\geq(\frac{8\mathrm{k}}{4\mathrm{k}-1})^2$ cannot be made small for any k.

For generic values of the parameters the expressions for the frequencies $\omega_{n,i}$ cannot be written in a useful form, but it is straightforward to determine their form in large ν (or, equivalently, large κ) expansion. The results are presented in appendix A.

The quartic equation $B_8=0$ leads to 4 solutions for ω_n^2. If all of them are non-negative, the solution is stable. Let us analyse the stability condition in the large κ limit. In this limit there are two different asymptotics of the different frequencies:

$$(i)\quad \omega_n^2 \to \frac{h_0}{4\kappa^2}+\frac{h_1}{\kappa^4}+\cdots\,, \qquad (ii)\quad \omega_n^2\to 4\kappa^2+g_0+\frac{g_1}{\kappa^2}+\cdots\,. \tag{2.37}$$

One finds (here we set $\mathrm{k}=1$ and $n\geq 0$):

$$h_0 = n^2[n^2+4-6\sin^2\gamma_0 \pm 2\sqrt{4n^2\cos^2\gamma_0-8\sin^2\gamma_0+9\sin^4\gamma_0}\,]\,. \tag{2.38}$$

The condition of non-negativity of h_0 is obtained at $n=1$, and is $\sin^2\gamma_0\leq\frac{3}{4}$. This implies the stability condition (1.7).

In general, for $\mathrm{k}=1,2,\ldots$ (this case is found by $n\to\frac{n}{\mathrm{k}}$ in (2.38)) the stability condition is obtained at $n=2\mathrm{k}-1$, and is[7]

$$\sin^2\gamma_0 \leq \frac{4\mathrm{k}-1}{4\mathrm{k}^2}\,. \tag{2.39}$$

The same stability condition was found above for $\nu=0$, and thus we expect it to be valid for any value of ν.

In the case (ii) one finds

$$\omega_n = 2\kappa+\frac{1}{2\kappa}[n^2+1-5\sin^2\gamma_0 \pm\sqrt{4n^2\cos^2\gamma_0+\sin^4\gamma_0}\,]+\cdots\,. \tag{2.40}$$

For comparison, a similar expansion of the frequencies $\sqrt{\kappa^2+n^2}$ in the AdS_5 directions in (2.15) is

$$\omega_n = \kappa+\frac{n^2}{2\kappa}-\frac{n^4}{8\kappa^3}+\cdots\,.$$

3. Fermionic part of the quadratic fluctuation action

Let us first recall the basic expressions for the quadratic fermionic action (see appendix B in [3]) and then find the corresponding spectrum of fluctuations. The quadratic part of the type-IIB $AdS_5\times S^5$ Green-Schwarz superstring action expanded near a particular bosonic string solution (with flat induced metric) is

$$L_F = i(\eta^{ab}\delta^{IJ}-\epsilon^{ab}s^{IJ})\bar\theta^I\varrho_a D_b\theta^J\,, \qquad \varrho_a\equiv\Gamma_A e^A_a\,, \qquad e^A_a\equiv E^A_M(X)\partial_a X^M\,, \tag{3.1}$$

[7] Note that the limiting case $\sin^2\gamma_0=3/4$ corresponds to the value of the angle $\gamma_0=\frac{\pi}{3}$ and may have some geometrical interpretation.

where $I, J = 1, 2$, s^{IJ} =diag(1,-1), ϱ_a are projections of the 10-d Dirac matrices and X^M are the string coordinates corresponding to the AdS_5 $(M = 0, 1, 2, 3, 4)$ and S^5 $(M = 5, 6, 7, 8, 9)$ factors. The covariant derivative D_a is

$$D_a\theta^I = \left(\delta^{IJ}\mathrm{D}_a - \frac{i}{2}\epsilon^{IJ}\Gamma_*\varrho_a\right)\theta^J\,, \qquad \Gamma_* \equiv i\Gamma_{01234}\,, \quad \Gamma_*^2 = 1\,, \tag{3.2}$$

$$\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}\,, \qquad \omega_a^{AB} \equiv \partial_a X^M \omega_M^{AB}\,. \tag{3.3}$$

Choosing the κ-symmetry gauge by equating the two Majorana-Weyl 10-d spinors,

$$\theta^1 = \theta^2 \equiv \theta\,, \tag{3.4}$$

we get

$$L_F = 2i\bar{\theta}D_F\theta\,, \qquad D_F = -\varrho^a\mathrm{D}_a - \frac{i}{2}\epsilon^{ab}\varrho_a\Gamma_*\varrho_b\,. \tag{3.5}$$

In the case of the S^5 solution (1.2) with k = 1 we shall label the tangent space coordinates by $A = 0, 5, 6, 7, 8, 9$ corresponding to the t direction of AdS_5 and $\gamma, \varphi_1, \psi, \varphi_2, \varphi_3$ directions of S^5. Then [3]

$$\varrho_0 = \kappa\Gamma_0 + \nu\cos\gamma_0\ \Gamma_6 + \mathrm{w}\sin\gamma_0\ \tilde{\Gamma}_8\,, \qquad \varrho_1 = \sin\gamma_0\ \Gamma_7\,, \qquad \varrho_{(a}\varrho_{b)} = \sin^2\gamma_0\ \eta_{ab}\,, \tag{3.6}$$

$$\tilde{\Gamma}_8 \equiv \cos\sigma\ \Gamma_8 + \sin\sigma\ \Gamma_9\,, \qquad \tilde{\Gamma}_9 \equiv \cos\sigma\ \Gamma_9 - \sin\sigma\ \Gamma_8\,. \tag{3.7}$$

The projected Lorentz connection has the following non-zero components

$$\begin{aligned} \omega_0^{65} &= -\nu\sin\gamma_0\,, \qquad \omega_0^{85} = \mathrm{w}\cos\gamma_0\ \cos\sigma\,, \qquad \omega_0^{95} = \mathrm{w}\cos\gamma_0\ \sin\sigma\,, \\ \omega_0^{87} &= -\mathrm{w}\sin\sigma\,, \qquad \omega_0^{97} = \mathrm{w}\cos\sigma\,, \qquad \omega_1^{75} = \cos\gamma_0\,. \end{aligned} \tag{3.8}$$

To eliminate the σ dependence we shall first do a local rotation in the 89-plane:

$$\theta = S(\sigma)\tilde{\theta}\,, \qquad S = e^{-\frac{1}{2}\sigma\Gamma_{89}}\,, \qquad S^{-1}\tilde{\Gamma}_i S = \Gamma_i\,, \quad i = 8, 9\,. \tag{3.9}$$

As a result, $\tilde{\theta}$ will be *anti*periodic in σ.

Then we get for the fermionic operator in (3.5)

$$\begin{aligned} D_F =& (\kappa\Gamma_0 + \nu\cos\gamma_0\ \Gamma_6 + \mathrm{w}\sin\gamma_0\ \Gamma_8)\times \\ &\times\left(\partial_\tau - \frac{1}{2}\nu\sin\gamma_0\ \Gamma_{65} + \frac{1}{2}\mathrm{w}\cos\gamma_0\ \Gamma_{85} + \frac{1}{2}\mathrm{w}\Gamma_{97}\right) - \\ &- \sin\gamma_0\ \Gamma_7(\partial_\sigma + \frac{1}{2}\cos\gamma_0\ \Gamma_{75} - \frac{1}{2}\Gamma_{89}) + \sin\gamma_0\Gamma_7(\nu\cos\gamma_0\ \Gamma_6 + \mathrm{w}\sin\gamma_0\ \Gamma_8)\Gamma_{01234}\,. \end{aligned} \tag{3.10}$$

We can put (3.10) in the form

$$\begin{aligned} D_F =& (\kappa\Gamma_0 + \nu\cos\gamma_0\ \Gamma_6 + \mathrm{w}\sin\gamma_0\ \Gamma_8)\partial_\tau - \sin\gamma_0\Gamma_7\partial_\sigma + \\ &+\frac{1}{2}\Gamma_5(-\nu\kappa\sin\gamma_0\ \Gamma_{06} + \mathrm{w}\kappa\cos\gamma_0\ \Gamma_{08} + \nu\mathrm{w}\Gamma_{68}) + \\ &+\frac{1}{2}[\kappa\mathrm{w}\Gamma_0 + \nu\mathrm{w}\cos\gamma_0\ \Gamma_6 + (\mathrm{w}^2+1)\sin\gamma_0\ \Gamma_8]\Gamma_{79} - \\ &- \sin\gamma_0(\nu\cos\gamma_0\ \Gamma_6 + \mathrm{w}\sin\gamma_0\ \Gamma_8)\Gamma_{07}\Gamma_{1234}\,. \end{aligned} \tag{3.11}$$

This can be simplified further by making two constant rotations in the 68 and 06 planes to transform $\kappa\Gamma_0 + \nu\cos\gamma_0\ \Gamma_6 + \mathrm{w}\sin\gamma_0\ \Gamma_8$ into $\sin\gamma_0\Gamma_0$:

$$\tilde{\theta} = S_{68}S_{06}\Psi\,, \qquad S_{68} = e^{-\frac{1}{2}p\Gamma_6\Gamma_8}\,, \qquad S_{06} = e^{-\frac{1}{2}q\Gamma_0\Gamma_8}\,, \tag{3.12}$$

$$\cos p = \frac{\nu}{a}\cos\gamma_0\,, \qquad \sin p = \frac{\mathrm{w}}{a}\sin\gamma_0\,, \qquad a \equiv \sqrt{\nu^2 + \sin^2\gamma_0}\,, \tag{3.13}$$

$$\cosh q = \frac{\kappa}{\sin\gamma_0}\,, \qquad \sinh q = \frac{a}{\sin\gamma_0}\,. \tag{3.14}$$

Then rescaling Ψ by $(\sin\gamma_0)^{1/2}$ we finish with the following fermionic lagrangian

$$\begin{aligned} L_F = 2i\bar{\Psi}\Big[&-\Gamma_0\left(\partial_0 + \frac{\kappa\cos\gamma_0}{2a}\Gamma_{65} + \frac{\mathrm{w}\nu}{2a}\Gamma_{85}\right) + \\ &+\Gamma_7\left(\partial_1 - \frac{\kappa\mathrm{w}}{2a}\Gamma_{69} - \frac{\nu\cos\gamma_0}{2a}\Gamma_{89}\right) - a\Gamma_{07}\Gamma_6\Gamma_{1234}\Big]\Psi\,, \end{aligned} \tag{3.15}$$

or

$$L_F = 2i\bar{\Psi}\left[\tau^0(\partial_0 + A_0) + \tau^1(\partial_1 + A_1) - a\tau_3\Gamma_6\Pi\right]\Psi\,, \tag{3.16}$$

where

$$\tau_a \equiv (\Gamma_0, \Gamma_7)\,, \qquad \tau_3 = \tau_0\tau_1\,, \qquad \Pi = \Gamma_{1234}\,, \qquad \Pi^2 = I\,, \tag{3.17}$$

$$A_0 = \frac{1}{2a}(\kappa\cos\gamma_0\Gamma_6 + \mathrm{w}\nu\Gamma_8)\Gamma_5\,, \qquad A_1 = -\frac{1}{2a}(\kappa\mathrm{w}\Gamma_6 + \nu\cos\gamma_0\Gamma_8)\Gamma_9\,. \tag{3.18}$$

This action may be interpreted as describing a collection of eight standard 2-d massive Majorana fermions on a *flat* 2-d background coupled to a constant non-abelian 2-d gauge field A_0, A_1. We may also split the fermions into the eigen-states of the projector $P = \frac{1}{2}(I + \Gamma_{1234})$ (which commutes with the rest of the operator). If one chooses a representation for Γ_A where Γ_0 and Γ_7 are 2-d Dirac matrices times a unit 8×8 matrix one gets 4+4 species of 2-d Majorana fermions with masses $\pm a = \pm\sqrt{\nu^2 + \sin^2\gamma_0}$.

Note that in the large ν limit $\kappa, \mathrm{w}, a \to \nu$, i.e.

$$A_0 \to \frac{1}{2}\nu\Gamma_8\Gamma_5\,, \qquad A_1 \to -\frac{1}{2}\nu\Gamma_6\Gamma_9\,, \tag{3.19}$$

so the action (3.16) simplifies.

While the presence of the τ_3 mass term in (3.15) has its origin in the coupling of the GS fermions to the 5-form background [10], the presence of the A_a connection term in (3.16) may be also interpreted as been due to the coupling to an effective NS-NS background in (2.25). For example, the coupling of the GS fermions to H_{MNK} field strength is through the term in the covariant derivative $D_a\theta^{1,2} = (\partial_a \pm \frac{1}{8}\partial_a X^K H_{KMN}\Gamma^{MN} + \cdots)\theta^{1,2}$. In the gauge (3.4) the H_{KMN} term contributes through the ϵ^{ab} term in (3.5), i.e. we get $\sim \bar{\theta}\epsilon^{ab}\varrho_a\partial_b X^K H_{KMN}\Gamma^{MN}\theta$. For $X^K \to u = \tau$ we get extra term $\sim \bar{\theta}\varrho_1 H_{uMN}\Gamma^{MN}\theta$ which for the background in (2.25) produces terms in A_1 in (3.16). Thus expanding near a circular classical string solution induces an effective H_{MNK} background in both the bosonic and the fermionic fluctuation sectors.

To solve the Dirac equation corresponding to (3.15) one should recall that while the original GS spinor variable θ in (3.1) was periodic on the 2-d cylinder, the rotated fermion

$\tilde{\theta}$ in (3.9) and thus Ψ in (3.12) is then anti-periodic, i.e. $\Psi(\tau, \sigma + 2\pi) = -\Psi(\tau, \sigma)$. That means one should look for solutions in the form

$$\Psi = \sum_{r \in Z + \frac{1}{2}} \sum_{i=1}^{8} \psi_r^{(i)} \ e^{i(\omega_{r,i}\tau \ + \ r\sigma)} \ . \tag{3.20}$$

The frequences ω_r can be found by solving the characteristic equation $F_8(\omega_r) = 0$. The latter can be obtained by multiplying the Dirac operator in (3.16) by its appropriate "conjugations" or by using an explicit representation for 6 Dirac matrices $\Gamma_0, \Gamma_5, \ldots, \Gamma_9$[8] (here $\Omega = \omega_r^2$, $\mathrm{k} = 1$, cf. (2.29))

$$\begin{aligned}
F_8 \ = \ & \Omega^4 - (5 + 4\ r^2 + 6\ \nu^2 - \cos 2\gamma_0)\ \Omega^3 + \\
& + \Big[6 + 6\ r^4 + 14\ \nu^2 + 9\ \nu^4 + 2\ r^2(2 + 7\ \nu^2) + \\
& \quad + 2(1 + 5\ r^2 + 2\ \nu^2)\ \sin^2\gamma_0 + \frac{3}{2}\sin^4\gamma_0 \Big] \Omega^2 - \\
& -2 \Big[2 + 2\ r^6 + 5\ \nu^2 + 5\ \nu^4 + 2\ \nu^6 + r^4\ (-2 + 5\ \nu^2) + r^2(-2 - 2\ \nu^2 + 5\ \nu^4) + \\
& \quad + \left[-1 + 7\ r^4 + \nu^4 + 2r^2(1 + 6\ \nu^2)\right] \sin^2\gamma_0 - \\
& \quad - \frac{1}{2}\left(1 - 7\ r^2 + \frac{1}{2}\nu^2\right) \sin^4\gamma_0 + \frac{1}{4}\sin^6\gamma_0 \Big] \Omega \\
& + (r^2 - 1)^2 \left[r^4 + 2r^2(\nu^2 - 1) + (\nu^2 + 1)^2 \right] + \\
& + 2(r^2 - 1)[3\ r^4 + 2\ r^2(\nu^2 - 2) + (1 + \nu^2)^2] \sin^2\gamma_0 \\
& + \frac{1}{2}[3 + 19\ r^4 + 5\ \nu^2 + 2\ \nu^4 + r^2\ (-14 + 5\ \nu^2)] \sin^4\gamma_0 + \\
& + \frac{1}{2}(-1 + 3\ r^2 - \nu^2)\ \sin^6\gamma_0 + \frac{1}{16}\sin^8\gamma_0 \, .
\end{aligned} \tag{3.21}$$

As in the bosonic case, while one cannot write down simple expressions for the frequencies in general, one may expand ω_r at large ν. The results are presented in appendix B.

The expressions for the frequencies take simple form when $\nu = 0$ (there is 4+4 degeneracy):

$$\nu = 0 \ : \qquad \omega_r^2 = r^2 + 1 + \frac{1}{2}\sin^2\gamma_0 \pm \sqrt{2[(2 - \sin^2\gamma_0)r^2 + \sin^2\gamma_0]} \, . \tag{3.22}$$

Another special case is the one of the unstable solution (2.32) with $\gamma_0 = \frac{\pi}{2}$ where we find

$$\gamma_0 = \frac{\pi}{2} \ : \qquad \omega_r = \pm\frac{1}{2}\left[\nu \pm \sqrt{\nu^2 + 2} \pm 2\sqrt{r^2 + \nu^2 + 1} \right] . \tag{3.23}$$

Let us mention also that to establish the connection to the point-like (BMN) case we need first to restore the dependence of F_8 in (3.21) on the discrete parameter k (which can be

[8] Since we only need their algebraic relations these Γ_m can be taken as Dirac matrices in 6 dimensions.

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done by the rescaling $r \to r/\mathrm{k}$, $\nu \to \nu/\mathrm{k}$, $\Omega \to \Omega/\mathrm{k}^2$, $F_8 \to \mathrm{k}^8 F_8$) and then send k. As a result, we find that F_8 becomes (cf. (2.30))

$$\mathrm{k} = 0 : \qquad F_8 = (r^2 + \nu^2 - \omega_r^2)^2 (r^2 - 2\nu\omega_r - \omega_r^2)(r^2 + 2\nu\omega_r - \omega_r^2) \,. \tag{3.24}$$

Here one is also to replace r by n taking integer values, since in the $\mathrm{k} = 0$ limit there is no local rotation (3.9) of the fermions that changes their periodicity in σ. Thus, as for the bosonic fluctuations (2.31), the roots of $F_8 = 0$ are indeed the same (up to a τ-dependent rotation contribution) as the "plane-wave" frequencies.

4. One-loop string sigma model correction to the energy

In this section we use the bosonic and fermionic spectrum to compute the one-loop sigma model correction to the energy of the solution. As in the static gauge $t = \kappa\tau$ in [4], the space-time energy and the 2-d energy (sum of $\frac{1}{2}\omega$ for all oscillator frequencies) are related by

$$E = \frac{1}{\kappa} \mathrm{E}_{2-d} \,. \tag{4.1}$$

Then the 1-loop correction is given by the standard sum of the oscillator frequencies sums

$$E = E_0 + E_1 + \cdots \,, \qquad E_1 = \frac{1}{2\kappa} \left(\sum_{n \in Z} \omega_n^B - \sum_{r \in Z + \frac{1}{2}} \omega_r^F \right) , \tag{4.2}$$

where ω_n^B and ω_r^F are bosonic and fermionic contributions, respectively:

$$\omega_n^B = \sum_{i=1}^{8} \omega_{n,i}^B \,, \qquad \omega_r^F = \sum_{i=1}^{8} \omega_{r,i}^F \,, \tag{4.3}$$

where the index i labels the characteristic frequencies. This expression is UV finite, as one can show directly from the expression for the total fluctuation lagrangian (2.15), (2.20), (3.16), or from the large-N and large r expansions of the frequencies given in appendices A and B (see also [3]).

One can check that the 1-loop correction vanishes in the "point-particle" limit $\mathrm{k} = 0$, in agreement with the non-renormalization of the energy of this BPS state dual to a gauge theory operator with protected conformal dimension [1]. In what follows we shall set $\mathrm{k} = 1$.

We would like to compute (4.2) as an expansion in $\frac{1}{\kappa}$ in the large κ limit. In the large κ limit there will be also exponentially small terms which we shall disregard. To estimate the value of the sums we shall approximate them by integrals as explained in appendix C.

As discussed in appendices A and B, the bosonic and fermionic frequencies $\omega_{n,i}^B$ and $\omega_{r,i}^F$ admit the following large κ expansion

$$\omega_{n,i}^B = \kappa\, \alpha_{-1,i}^B \left(\frac{n}{\kappa}\right) + \alpha_{0,i}^B \left(\frac{n}{\kappa}\right) + \frac{1}{\kappa} \alpha_{1,i}^B \left(\frac{n}{\kappa}\right) + \cdots \,, \tag{4.4}$$

$$\omega_{r,i}^F = \kappa\, \alpha_{-1,i}^F \left(\frac{r}{\kappa}\right) + \alpha_{0,i}^F \left(\frac{r}{\kappa}\right) + \frac{1}{\kappa} \alpha_{1,i}^F \left(\frac{r}{\kappa}\right) + \cdots \,, \tag{4.5}$$

where we keep n/κ and r/κ fixed in the expansion. The values $\alpha_{a,i}(\frac{m}{\kappa})$ can be considered as values of the functions $\alpha_{a,i}(x)$ at points $x_m = m/\kappa$, where $m \in Z$ for bosons, and $m \in Z + \frac{1}{2}$ for fermions, and $\Delta \equiv x_{m+1} - x_m = 1/\kappa$.

We also need to regularize the bosonic and fermionic sums. This can be done by multiplying each term in the sums by, e.g. $\mathrm{e}^{-|x_m|\epsilon}$. Since the fermionic functions $\alpha^F_{a,i}(x)$ are smooth for all values of x, the sums over $r \in Z + \frac{1}{2}$ are replaced by integrals from $-\infty$ to $+\infty$. However, not all of the bosonic functions $\alpha^B_{a,i}(x)$ are smooth at $x = -\frac{2}{\kappa}, -\frac{1}{\kappa}, 0, \frac{1}{\kappa}, \frac{2}{\kappa}$. Therefore, we obtain the following formula for the bosonic contribution in (4.2)

$$\begin{aligned} \frac{1}{2\kappa}\sum_{n\in Z}\omega^B_n &= \frac{1}{2\kappa}\left(\omega^B_0 + 2\omega^B_1 + 2\omega^B_2\right) - \frac{1}{2}\int_{-2/\kappa}^{3/\kappa} dx\left[\kappa\,\alpha^B_{-1}(x) + \frac{1}{\kappa}\alpha^B_1(x) + \cdots\right] \\ &+\frac{1}{2}\int_{-\infty}^{\infty} dx\left[\kappa\,\alpha^B_{-1}(x) + \frac{1}{\kappa}\alpha^B_1(x) + \cdots\right]. \end{aligned} \tag{4.6}$$

Here we have taken into account that $\omega^B_{-n} = \omega^B_n$, $\alpha^B_0 = \alpha^F_0 = 0$, and used the result of appendix C to replace the two sums, $\sum_{-\infty}^{-3}$ and $\sum_3^{\infty}$, by the integrals.

As can be shown by a straightforward computation, the functions α^B_{-1} and α^F_{-1}, and α^B_1 and α^F_1, are equal to each other and are given by

$$\alpha^B_{-1}(x) = \alpha^F_{-1}(x) = 8\sqrt{x^2+1}\,, \tag{4.7}$$

$$\alpha^B_1(x) = \alpha^F_1(x) = -\frac{2\left(-1 + 3\sin^2\gamma_0 + 2\,x^2\sin^2\gamma_0\right)}{\sqrt{(x^2+1)^3}}. \tag{4.8}$$

Therefore, the bosonic and fermionic integrals from $-\infty$ to $+\infty$ cancel each other up to the order $1/\kappa^2$, and the correction to the energy is given by the first line in (4.6).

To compute the correction we need to know the large κ expansions of the bosonic frequencies at fixed n (here we set k $= 1$). They are given up to the order $\frac{1}{\kappa}$ by the formulas (A.4)–(A.8) of appendix A. Using them, we get the following result

$$E_1 = \frac{1}{\kappa^2}e_1(\gamma_0)\,, \tag{4.9}$$

$$e_1(\gamma_0) \equiv \frac{1}{2}\left[5\sin^2\gamma_0 - 9 + \sqrt{9 - 12\sin^2\gamma_0} + 4\sqrt{4 - 3\sin^2\gamma_0}\right]. \tag{4.10}$$

Taking into account that at large κ

$$\frac{1}{\kappa^2} = \frac{\lambda}{(J+2J')^2} + \cdots\,, \tag{4.11}$$

we can rewrite (4.9) in the following form

$$E_1 = \frac{\lambda}{(J+2J')^2}e_1(\gamma_0) + \cdots\,. \tag{4.12}$$

At large κ we can also express $\sin^2\gamma_0$ through the angular momenta J and J'

$$\sin^2\gamma_0 \approx \frac{2J'}{J+2J'} \le \frac{3}{4}\,. \tag{4.13}$$

At small values of γ_0 or $J' \ll J$ we thus get $e_1(\gamma_0) \approx 1$, i.e.[9]

$$E_1 \approx \frac{\lambda}{(J+2J')^2} \, . \tag{4.14}$$

Combining (4.9) with the classical result for the energy (1.10), we obtain the following expression (cf. (1.18))

$$E = J + 2J' + \frac{\lambda}{(J+2J')^2}[J' + e_1(\gamma_0) + \cdots] \, . \tag{4.15}$$

We conclude that the term $J+2J'$ is not modified by the one-loop sigma model correction. As was discussed above, $J' \sim \sqrt{\lambda}$ is large in the semiclassical approximation, and, therefore, the one-loop sigma model correction is subleading at least at the first order in λ. Since the correction admits an asymptotic expansion in $\frac{1}{\kappa^{2m}}$ with coefficients depending only on $\sin^2\gamma_0$, the one-loop sigma model correction is also subleading at any order in λ.

It is tempting then to conjecture that all higher-loop sigma model string corrections are also subleading at large J', and, therefore, in this regime the classical formula for the energy (1.10) is exact to all orders in λ. It should then be true also at weak coupling and thus should represent a prediction for the corresponding anomalous dimensions on the SYM side. It should be possible to check it using the methods of [11, 12, 13].

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A. Bosonic frequencies

Bosonic spectrum of physical fluctuations is determined by zeros of the determinant (2.29) of the characteristic matrix. There are also 4 bosons with masses $m^2 = \kappa^2$ coming (2.15) from the AdS_5 part of the background. In what follows k will be set to 1. The dependence on k can be restored by rescaling $n \to n/\mathrm{k}$, $\nu \to \nu/\mathrm{k}$, $\omega_n \to \omega_n/\mathrm{k}$.

[9]Note that the correction thus does not vanish for $J' = 0$. This may look as contradicting to the fact that at $\gamma_0 = 0$ our solution should be representing a BPS state — a point particle rotating along a big circle of S^5. As already mentioned above, to recover the point-like case one should actually set k to 0, while (4.9) was derived assuming $\mathrm{k} = 1$. In general, the $\gamma_0 \to 0$ limit is subtle: expansion near a point-like string is not a limit of expansion near an extended string. This is clear from a comparison of the fluctuation lagrangians in the two cases (cf. (2.20)). A smooth limit is found by keeping γ_0 arbitrary while sending $\mathrm{k} = 0$: in this case we are dealing with "off-diagonal" plane of rotation of a point-like string and both J and J' are non-zero.

A.1 Expansion at large-N

To check the ultra-violet finiteness of the model we need to know the large-N expansion of the frequencies up to the order $1/n$. There are 4 different non-negative frequencies corresponding to 4 choices of the signs of the two square roots in the frequency below

$$\omega_n = |n| \pm \sqrt{2+\nu^2 \pm \sqrt{(1+\cos^2\gamma_0)^2+4\nu^2\cos^2\gamma_0}} + \frac{\nu^2}{2|n|} + \cdots\,, \qquad |n| \gg \nu\,, \tag{A.1}$$

and 4 frequencies of the AdS_5 fluctuations ($\kappa^2 = \nu^2 + 2\sin^2\gamma_0$)

$$\omega_n^{AdS} = |n| + \frac{\kappa^2}{2|n|} + \cdots\,, \qquad |n| \gg \nu\,. \tag{A.2}$$

Summing up the contributions of these 4+4=8 frequencies, we get

$$\sum_{i=1}^{8} \omega^B_{n,i} = 8|n| + 4\,\frac{\nu^2+\sin^2\gamma_0}{|n|} + \cdots\,. \tag{A.3}$$

A.2 Expansion at large κ and fixed n

To compute the one-loop correction to the vacuum energy we need the expansion of the frequencies at large κ (or, equivalently, large ν) and fixed n. It is given, up to the order $1/\kappa$, by the following expressions

$$\begin{aligned}
\omega^B_{n,1}, \omega^B_{n,2} &= \frac{1}{\kappa}\frac{|n|}{2}\sqrt{4+n^2-6\,\sin^2\gamma_0 \pm 2\sqrt{4\,n^2\cos^2\gamma_0 - 8\,\sin^2\gamma_0 + 9\sin^4\gamma_0}}\,, && \text{(A.4)}\\
\omega^B_{n,3}, \omega^B_{n,4} &= 2\,\kappa + \frac{1}{2\,\kappa}\left(2+n^2-5\,\sin^2\gamma_0 \pm \sqrt{4\,n^2\cos^2\gamma_0+\sin^4\gamma_0}\right)\,, && \text{(A.5)}\\
\omega^B_{n,i} &= \kappa + \frac{n^2}{2\kappa}\,, \qquad i = 5,6,7,8\,. && \text{(A.6)}
\end{aligned}$$

A.3 Expansion at large κ and fixed n/κ

The one-loop computation also requires the knowledge of the expansion at large κ and fixed $\frac{n}{\kappa}$. Introducing $x = \frac{n}{\kappa}$, and keeping it fixed, we obtain the following expansion for $\kappa \gg 1$

$$\begin{aligned}
\omega^B_{n,1}, \omega^B_{n,2} &= \kappa\left(1+\sqrt{x^2+1}\right) \pm \frac{|x|\cos\gamma_0}{\sqrt{x^2+1}} + \frac{1}{\kappa}\left[\frac{1}{2}\cos 2\gamma_0 - \frac{2x^2\sin^2\gamma_0+3\sin^2\gamma_0-1}{2\sqrt{(x^2+1)^3}}\right], && \text{(A.7)}\\
\omega^B_{n,3}, \omega^B_{n,4} &= \kappa\left(-1+\sqrt{x^2+1}\right) \pm \frac{|x|\cos\gamma_0}{\sqrt{x^2+1}} + \frac{1}{\kappa}\left[-\frac{1}{2}\cos 2\gamma_0 - \frac{2x^2\sin^2\gamma_0+3\sin^2\gamma_0-1}{2\sqrt{(x^2+1)^3}}\right], && \text{(A.8)}\\
\omega^B_{n,i} &= \frac{1}{\kappa}\sqrt{x^2+1}\,, \qquad i = 5,6,7,8\,. && \text{(A.9)}
\end{aligned}$$

Summing up all the 8 frequencies, we get

$$\sum_{i=1}^{8} \omega^B_{n,i} = 8\kappa\sqrt{x^2+1} - \frac{1}{\kappa}\,\frac{2(2x^2\sin^2\gamma_0+3\sin^2\gamma_0-1)}{\sqrt{(x^2+1)^3}} + \cdots\,. \tag{A.10}$$

B. Fermionic frequencies

Fermionic spectrum of physical fluctuations is determined (for k = 1) by zeros of the determinant (3.21) of the characteristic matrix. The dependence on k can be restored again by rescaling $r \to r/\mathrm{k}$, $\nu \to \nu/\mathrm{k}$, $\omega_r \to \omega_r/\mathrm{k}$.

B.1 Expansion at large r

To check that the UV divergences coming from the bosonic sector (cf. (A.3)) and cancelled by the fermions we need to know the large r expansion of the frequencies up to the order $1/r$. Among the 8 fermionic frequencies there are only 4 different corresponding to 4 choices of the signs of the 2 square roots in the expression ($r = \pm\frac{1}{2}, \ldots$)

$$\omega_r = |r| \pm \sqrt{1 + \nu^2 + \cos^2\gamma_0 \pm \nu\sqrt{\nu^2 + 2\sin^2\gamma_0}} + \frac{\nu^2 + \sin^2\gamma_0}{2|r|} + \cdots, \qquad |r| \gg \nu\,. \tag{B.1}$$

Summing up these 8 frequencies, we get

$$\sum_{i=1}^{8} \omega^F_{r,i} = 8|r| + 4\,\frac{\nu^2 + \sin^2\gamma_0}{|r|} + \cdots. \tag{B.2}$$

Comparing this expression with the sum of the bosonic contributions (A.3), we find that indeed the 2-d UV cancel in the 1-loop correction to the energy (4.2).

B.2 Expansion at large κ and fixed r

Even though we do not need the expansion of the fermionic frequencies at large κ and fixed r to compute the one-loop correction to the vacuum energy, we present here this expansion for completeness

$$\omega^F_{r,1} = \frac{1}{2\kappa}|r^2 - \cos^2\gamma_0|\,, \tag{B.3}$$

$$\omega^F_{r,2}, \omega^F_{r,3} = \kappa + \frac{1}{2\kappa}\left(r^2 + \cos 2\gamma_0 \ \pm \sqrt{4\,n^2\cos^2\gamma_0 + \sin^4\gamma_0}\right), \tag{B.4}$$

$$\omega^F_{r,4} = 2\,\kappa + \frac{1}{2\,\kappa}\left(1 + r^2 - 3\,\sin^2\gamma_0\right). \tag{B.5}$$

B.3 Expansion at large κ and fixed r/κ

The one-loop computation requires knowledge of the expansion of the fermionic frequencies at large κ and fixed r/κ. Introducing $x = r/\kappa$, and keeping it fixed, we obtain the following expansion

$$\omega^F_{r,1}, \omega^F_{r,2} = \kappa\sqrt{x^2+1} \pm \frac{|x|\cos\gamma_0}{\sqrt{x^2+1}} - \frac{1}{\kappa}\,\frac{x^2\sin^2\gamma_0 + 2\sin^2\gamma_0 - 1}{2\sqrt{(x^2+1)^3}}, \tag{B.6}$$

$$\omega^F_{r,3}, \omega^F_{r,4} = \kappa\left(\pm 1 + \sqrt{x^2+1}\right) + \frac{1}{\kappa}\left(\pm\frac{1}{2}\cos 2\gamma_0 \ - \frac{\sin^2\gamma_0}{2\sqrt{x^2+1}}\right). \tag{B.7}$$

Summing up all the eight fermionic frequencies, we get

$$\sum_{i=1}^{8} \omega^F_{r,i} = 8\kappa\sqrt{x^2+1} - \frac{1}{\kappa}\,\frac{2(2x^2\sin^2\gamma_0 + 3\sin^2\gamma_0 - 1)}{\sqrt{(x^2+1)^3}} + \cdots. \tag{B.8}$$

Remarkably, this expression coincides with the sum of the bosonic frequencies (A.10).

C. Approximation of an infinite sum by an integral

Let us recall how sums of the form (4.2) can be replaced by integrals. Assume that we are given a smooth function $f(x)$ and its values at points x_i, $i = 1, \cdots, N$; $x_{i+1} - x_i = \Delta$. We are to find a function $g(x)$ such that the following formula is valid

$$\sum_{i=1}^{N} f(x_i) = \frac{1}{\Delta} \int_{x_1}^{x_{N+1}} dx\; g(x) + O(\Delta^5) = \frac{1}{\Delta} \sum_{i=1}^{N} \int_{x_i}^{x_{i+1}} dx\; g(x) + O(\Delta^5)\,. \tag{C.1}$$

We see that $g(x)$ should satisfy

$$f(x_i) = \int_{x_i}^{x_{i+1}} dx\; g(x) = g_i + \frac{\Delta}{2} g_i' + \frac{\Delta^2}{6} g_i'' + \frac{\Delta^3}{24} g_i^{(3)} + \frac{\Delta^4}{120} g_i^{(4)} + O(\Delta^5)\,,$$

where $g_i = g(x_i)$, $g_i' = \frac{d}{dx} g(x_i)$, $g_i^{(k)} = \frac{d^k}{dx^k} g(x_i)$, and so on. It is not difficult to check that this formula will be fulfilled if we choose $g(x)$ to be

$$g(x) = -\frac{2}{15} f(x - \Delta) + \frac{6}{5} f\left(x - \frac{\Delta}{2}\right) + \frac{1}{30} f(x) - \frac{2}{15} f\left(x + \frac{\Delta}{2}\right) + \frac{1}{30} f(x + \Delta)\,. \tag{C.2}$$

If we are interested in computing the sum in (C.1) only up to the order Δ^2 then a simpler formula can be used

$$g(x) = \frac{1}{3} f(x - \Delta) + \frac{5}{6} f(x) - \frac{1}{6} f(x + \Delta)\,. \tag{C.3}$$

Note that to use these expressions, the function $f(x)$ has to be smooth in the interval $[x_1, x_N]$.

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